PHY456 Lecture 6:

Addition of Angular Momenta (part I)

Review:

Up until now, we've found that rotations in space are performed with unitary operators

$$^{(r)}\hat{u}(\alpha, \vec{n}) = e^{-i \frac{\alpha}{\hbar} \vec{n} \cdot \vec{L}} \qquad [SO(3)]$$

$$^{(\frac{1}{2})}\hat{u}(\alpha, \vec{n}) = e^{-i \frac{\alpha}{\hbar} \vec{n} \cdot \hat{\vec{S}}} \qquad [SU(2)]$$

The "generators" of rotations are the orbital angular momentum operators $\hat{L}_i$ or the spin ones $\hat{S}_i$.

They obey the algebra $[\hat{L}_i, \hat{L}_j] = i \epsilon_{ijk} \hbar \hat{L}_k$

$$[\hat{S}_i, \hat{S}_j] = i \epsilon_{ijk} \hbar \hat{S}_k$$

and have

$\hat{\vec{L}}^2$, $\hat{L}_z$ eigenstate $|\ell, m\rangle$: $m = -\ell, -\ell+1, \dots, \ell-1, \ell$

$$\vec{L}^2 |\ell, m\rangle = \ell(\ell+1)\hbar^2 |\ell, m\rangle$$

$$\hat{L}_z |\ell, m\rangle = m\hbar |\ell, m\rangle$$

Pauli matrices

$\sigma_0 = \mathbb{1}$

$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y$$

$$L_\pm |\ell, m\rangle = \sqrt{\ell(\ell+1) - m(m\pm 1)}\, \hbar\, |\ell, m\pm 1\rangle$$

and $\hat{\vec{S}}^2, \hat{S}_z$ eigenstates $|s, s_z\rangle$

$$\hat{\vec{S}}^2 |s, s_z\rangle = \frac{1}{2}\left(\frac{1}{2}+1\right) \hbar^2 |s, s_z\rangle$$

$$\hat{S}_z |s, s_z\rangle = s_z \hbar\, |s, s_z\rangle$$

$$\hat{S}_\pm = \hat{S}_x \pm i\hat{S}_y$$

$$\hat{S}_\pm |s, s_z\rangle = \sqrt{s(s+1) - s_z(s_z \pm 1)}\, \hbar\, |s, s_z\rangle$$

For $s = \frac{1}{2}$, there is a complete parallel between the $\hat{S}$ and $\hat{L}$ algebra.

What about $s \neq \frac{1}{2}$ states?

Let's introduce general notation $\hat{J}_i$; $[\hat{J}_i, \hat{J}_j] = i\epsilon_{ijk} \hbar \hat{J}_k$

with eigenstates $\{|j,m\rangle$, $j = \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots$ and $m = -j, \ldots, j\}$

$s=\frac{1}{2}$ $s=1$ $s=\frac{3}{2}, \frac{5}{2}$ will see how these come about

In other words, $\hat{J}_i$ can be either integral or half-integral angular momentum

$$\hat{\vec{J}}^2 |j,m\rangle = j(j+1)\hbar^2 |j,m\rangle$$

$$\hat{J}_z |j,m\rangle = m\hbar |j,m\rangle$$

$$\hat{J}_\pm |j,m\rangle = \sqrt{j(j+1) - m(m\pm 1)}\ \hbar\ |j,m\pm 1\rangle.$$

The unitary operator for angular momentum $\hat{\vec{J}}$ is then

$${}^{(j)}\mathcal{U}(\alpha, \vec{n}) = e^{-i\frac{\alpha}{\hbar}\vec{n}\cdot\hat{\vec{J}}} \qquad [SU(2)^{(j)}]$$

and the states $\{|j,m\rangle : m=-j,\ldots,j\}$ are rotated into each other.

There is no subset of $2j+1$ states w/ given j that is invariant under all rotations

(We say that the states $\{|j,j\rangle, |j,j-1\rangle, \dots, |j,-j+1\rangle, |j,-j\rangle\}$ form an "irreducible representation" of the rotation group $SU(2)$).

Addition Problem:

Consider two systems with angular momentum $\vec{J}_1$ and $\vec{J}_2$.

Let both $\vec{J}$'s act in separate Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$:

$$\{|j_1, m_1\rangle, \quad m_1 = -j_1, \dots, j_1\} \qquad 2j_1+1 \text{ - dimensional}$$

$$\{|j_2, m_2\rangle, \quad m_2 = -j_2, \dots, j_2\} \qquad 2j_2+1 \text{ - dimensional}$$

Consider the tensor product of the two Hilbert spaces, spanned by the vectors

$$\left\{ |j_1, m_1\rangle \otimes |j_2, m_2\rangle : \begin{array}{l} m_1 = -j_1, \dots, j_1 \\ m_2 = -j_2, \dots, j_2 \end{array} \right\} \qquad (2j_1+1)(2j_2+1) \text{ - dimensional.}$$

from which we can label the states $|j_1 m_1, j_2 m_2\rangle$.

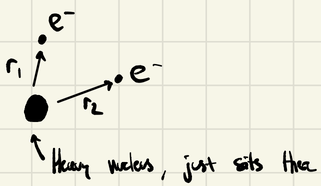
<u>Define</u>: $\hat{\vec{J}} = \hat{\vec{J}}_1 + \hat{\vec{J}}_2$, the operator of total angular momentum of the system ① ⊗ ②. $\vec{J}$ has eigenstates

$\{|j,m\rangle,\ m=-j,\dots,+j\}$ labelled by j, which, for each given j are $2j+1$ - dimensional.

Addition of angular momentum: which values of j occur in $\{|j_1 m_1, j_2 m_2\rangle,\ m_1=-j_1,\dots,j_1,\ \dots\}$? How is this basis related to the $\{|j,m\rangle,\ j=\dots,\ m=\dots\}$ basis?

Motivations for solving the problem:

1) Classical mechanics:



$\alpha>0$ attractive potential

Hamiltonian for e_1^-: $\hat{H}_1 = -\frac{\hbar^2\nabla_1^2}{2m} - \alpha\frac{1}{|\vec{r}_1|}$

and $[\hat{L}_1, \hat{H}_1]=0$ since $\frac{1}{|\vec{r}_1|}$ is spherically symmetric.

Hamiltonian for e_2^-: $\hat{H}_2 = -\frac{\hbar^2\nabla_2^2}{2m} - \alpha\frac{1}{|\vec{r}_2|}$

and $[\hat{L}_2, \hat{H}_2]=0$ for same reason.

Then for the total system $\hat{H}=\hat{H}_1+\hat{H}_2$, can label eigenstates $|j_1 m_1, j_2 m_2\rangle$ since $\hat{H}$ commutes with both $\hat{L}_1$ and $\hat{L}_2$.

However, the two electrons have mutual coulomb repulsion, so

$$\hat{H}' = \hat{H}_1 + \hat{H}_2 + \gamma \frac{1}{|\vec{r}_1 - \vec{r}_2|}$$

$$= \underbrace{-\frac{\hbar^2}{2m}\left(\nabla_1^2 + \nabla_2^2\right) - \alpha\left(\frac{1}{|\vec{r}_1|} + \frac{1}{|\vec{r}_2|}\right)}_{\text{invariant under independent rotations of } \vec{r}_1 \text{ and } \vec{r}_2} + \underbrace{\gamma \frac{1}{|\vec{r}_1 - \vec{r}_2|}}_{\text{requires simultaneous rotations of } \vec{r}_1 \text{ and } \vec{r}_2 \text{ to be invariant}}$$

Thus we must represent eigenstates as $|l, m\rangle$ not $|l_1 m_1, l_2 m_2\rangle$.

Solving this Hamiltonian is impossible. We can use various tricks (ie, perturbation theory) which all use products of Hydrogen atom wavefunctions $(\hat{H}_1, \hat{H}_2)$.

So we must learn, for a given $\{|l_1 m_1, l_2 m_2\rangle \; m_1 = -l_1, l_1, \; m_2 = -l_2, l_2\}$ how to find tensor product Hilbert space states $|l, m\rangle$.

2) Two-Spin System (ie. Ferromagnets)

→ Spin-Spin interactions in atoms.

We consider a system of two spin-$\frac{1}{2}$ particles $\hat{\vec{S}}_1$ and $\hat{\vec{S}}_2$.

4-dim Hilbert space : $|S_{1z}, S_{2z}\rangle$.

each $\pm\frac{\hbar}{2}$.

Let $\hat{H} = \alpha \hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2$ be a Hamiltonian, which commutes with $\hat{\vec{S}} = \hat{\vec{S}}_1 + \hat{\vec{S}}_2$ but not individually with $\hat{\vec{S}}_1$ and $\hat{\vec{S}}_2$.

What values can $\vec{J}$ (total $\vec{S}$) take?

Answer: see tutorial.

$S=1$: three states "triplet" spin-1 (3-fold degeneracy).

$S=0$: single state "singlet" spin-0.

$\hat{H} = \alpha \hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2 \Rightarrow$ $\alpha > 0$ singlet state has less energy

$\alpha < 0$ triplet state has less energy